Lec 11

Solving Linear Differential Eq:

The particular solution

I Example:

$$\ddot{y} - 3\dot{y} + 2y = f(t) \quad ✪$$

$$y(0) = 0, \quad \dot{y}(0) = 0.$$

Find one particular $y(t)$ which satisfies ✪.

Sol^n As before, we define

$$x_1 = y, \quad x_2 = \dot{y}$$

and obtain

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0, \; x_2(0) = 0$$

$$\dot{x}_2 = \ddot{y} = 3\dot{y} - 2y + f(t)$$

$$= 3x_2 - 2x_1 + f(t).$$

Define

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \text{we obtain}$$

$$\dot{\mathbf{X}} = A\,\mathbf{X} + b\,f(t) \qquad \textcircled{\triangle}$$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{X}(0) = 0$$

A particular solution of $\textcircled{\triangle}$ is given by

$$\boxed{\mathbf{X}(t) = \int_0^t e^{A(t-\tau)}\, b\, f(\tau)\, d\tau .}$$

From page 10·3, we have

$$e^{A(t-\tau)}b = \begin{pmatrix} e^{2(t-\tau)} - e^{(t-\tau)} \\ -e^{(t-\tau)} + 2e^{2(t-\tau)} \end{pmatrix}$$

Hence

$$x_1(t) = \int_0^t \left[e^{2(t-\tau)} - e^{(t-\tau)}\right] f(\tau)\, d\tau .$$

$$x_2(t) = \int_0^t \left[-e^{(t-\tau)} + 2e^{2(t-\tau)}\right] f(\tau)\, d\tau .$$

Finally

$$Y(t) = \int_0^t \left[e^{2(t-\tau)} - e^{(t-\tau)}\right] f(\tau)\, d\tau \tag{**}$$

If we define

$$h(t) = e^{2t} - e^{t}$$

we can rewrite (**) as

$$\boxed{Y(t) = \underbrace{\int_0^t h(t-\tau)\, f(\tau)\, d\tau}_{\text{Convolution of two functions } h(t),\ f(t).} .}$$

11·4

For various forcing function $f(t)$, we write down $y(t)$ as follows

a. $f(t) = \sin\omega t$

$$y(t) = \frac{3\omega\cos\omega t + (2-\omega^2)\sin\omega t + \omega(1+\omega^2)e^{2t} - \omega(4+\omega^2)e^{t}}{(4+\omega^2)(1+\omega^2)}$$

b. $f(t) = e^{-t}$

$$y(t) = e^{-t}\left(\frac{1}{6} + \frac{1}{3}e^{t} - \frac{1}{2}e^{2t}\right)$$

c. $f(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$ unit step function.

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{2t} - e^{t}$$

II Example :

$$\dddot{y} + 12\ddot{y} + 47\dot{y} + 60y = f(t) \quad \circledast$$

$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

Find one particular solution $y(t)$ which satisfies $\circledast$

Sol$^{\underline{n}}$ As in page 10·7, we define

$$\underline{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \qquad \underline{X}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \dot{X} &= AX + b\,f(t) \\ y(t) &= c\,X. \end{aligned}} \qquad (\star\star)$$

- Verify that (★) & (★★) are equivalent.

- Verify that (★★) has a particular solution given by

$$y(t) = \int_0^t c\,e^{A(t-\tau)}\,b\,f(\tau)\,d\tau.$$

Remark: We can compute $c\,e^{A(t-\tau)}b$ explicitly but since $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we conclude that $e^{A(t-\tau)}b$ is the last column of $e^{A(t-\tau)}$. Since $c = (1 \ 0 \ 0)$, we conclude that $c\,e^{A(t-\tau)}b$ is the first entry of the last column of $e^{A(t-\tau)}$.

From calculations on page 10·8 it follows that

$$c e^{A(t-\tau)} b =$$

$$\frac{1}{2} e^{-3(t-\tau)} - e^{-4(t-\tau)} + \frac{1}{2} e^{-5(t-\tau)}$$

Define

$$h(t) = c e^{At} b$$

$$= \frac{1}{2} e^{-3t} - e^{-4t} + \frac{1}{2} e^{-5t}$$

and write

$$\underbrace{y(t) = \int_0^t h(t-\tau) f(\tau) \, d\tau}_{\text{convolution of two functions } h(t) \,\&\, f(t) \text{ as in page 11·3.}}$$

(11·8)

For various forcing function $f(t)$, we write down $y(t)$ as follows:

a. $f(t) = \sin 3t$

$$y(t) = -\frac{38}{1275}\cos^3 t + \frac{19}{850}\cos t - \frac{16}{1275}\sin t\cos^2 t + \frac{4}{1275}\sin t + \frac{1}{12}e^{-3t} - \frac{3}{25}e^{-4t} + \frac{3}{68}e^{-5t}.$$

b. $f(t) = e^{-t}$

$$y(t) = \frac{1}{24}e^{-t} - \frac{1}{4}e^{-3t} + \frac{1}{3}e^{-4t} - \frac{1}{8}e^{-5t}$$

c. $f(t) = 1$

$$y(t) = \frac{1}{60} - \frac{1}{6}e^{-3t} + \frac{1}{4}e^{-4t} - \frac{1}{10}e^{-5t}$$

11·8a

d. $f(t) = \sin\omega t$

$$y(t) = \frac{1}{2}\,\frac{1}{(9+\omega^2)(16+\omega^2)(25+\omega^2)} \times$$

$$\Big[\left.\begin{matrix}(-94\omega+2\omega^3)\cos\omega t \\ +(120-24\omega^2)\sin\omega t\end{matrix}\right\} \Leftarrow \text{Forcing function.}$$

$$\left.\begin{matrix}+(400\omega+41\omega^3+\omega^5)e^{-3t} \\ +(-450\omega-68\omega^3-2\omega^5)e^{-4t} \\ +(144\omega+25\omega^3+\omega^5)e^{-5t}\end{matrix}\right\} \Leftarrow \text{Natural Modes.}$$

III Example :

$$\dot{x}_1 = \frac{1782}{1188} x_1 - \frac{990}{1188} x_2 - \frac{396}{1188} x_3 + 3f(t)$$

$$\dot{x}_2 = -\frac{6534}{1188} x_1 - \frac{8514}{1188} x_2 - \frac{4356}{1188} x_3 + 5f(t)$$

$$\dot{x}_3 = \frac{17226}{1188} x_1 + \frac{28710}{1188} x_2 + \frac{13860}{1188} x_3 + 9f(t)$$

$$x_1(0) = x_2(0) = x_3(0) = 0$$

$$y(t) = x_1(t)$$

— x —

Define

$$b = \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

B as in page 10·15.

We obtain

$$\dot{\underline{X}} = B\underline{X} + b f(t)$$
$$y = c\underline{X}$$

From calculation on page 10·15,

$$e^{Bt}b = \begin{pmatrix} -\frac{26}{3}te^{2t} + 3e^{2t} \\ -\frac{286}{3}te^{2t} + 5e^{2t} \\ \frac{754}{3}te^{2t} + 9e^{2t} \end{pmatrix}$$

It follows that $ce^{Bt}b = h(t) =$

$$-\frac{26}{3}te^{2t} + 3e^{2t} = \left(-\frac{26}{3}t + 3\right)e^{2t}$$

$$y(t) = \int_0^t \left(-\frac{26}{3}(t-\tau) + 3\right)e^{2(t-\tau)} f(\tau)\, d\tau$$

$\boxed{\text{Convolution Integral again}}$

Alternative approach :—

Let us define as in page 10·18, a new vector $z(t)$, as follows:

$$\underline{X}(t) = P\,z(t)$$

where P is the 3×3 matrix on page 10·18. It follows that

$$P\dot{z} = \underline{\dot{X}} = B\underline{X} + b\,f(t)$$

$$= BPz(t) + b\,f(t)$$

$$\Rightarrow \boxed{\begin{aligned} \dot{z} &= (P^{-1}BP)z(t) + P^{-1}b\,f(t) \\ Y &= C\underline{X} = CP\,z(t) \end{aligned}}$$

As an page 10·19, let us write

$$A = P^{-1}BP\,,\ b_1 = P^{-1}b,\ c_1 = CP$$

and obtain

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} -\frac{180}{1188} \\ \frac{3432}{1188} \\ \frac{960}{1188} \end{pmatrix} \leftarrow$$

From bottom of page 10·19

$$c_1 = \begin{pmatrix} -3 & 2 & -4 \end{pmatrix}$$

$$\boxed{\begin{aligned} \dot{Z} &= A Z + b_1 f(t) \\ y &= c_1 Z \end{aligned}}$$

$$h(t) =$$

$$\underbrace{\begin{pmatrix}-3 & 2 & -4\end{pmatrix}}_{c_1} \underbrace{\begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t}\end{pmatrix}}_{e^{At}} \underbrace{\begin{pmatrix} -\frac{180}{1188} \\ \frac{3432}{1188} \\ \frac{960}{1188}\end{pmatrix}}_{b_1}$$

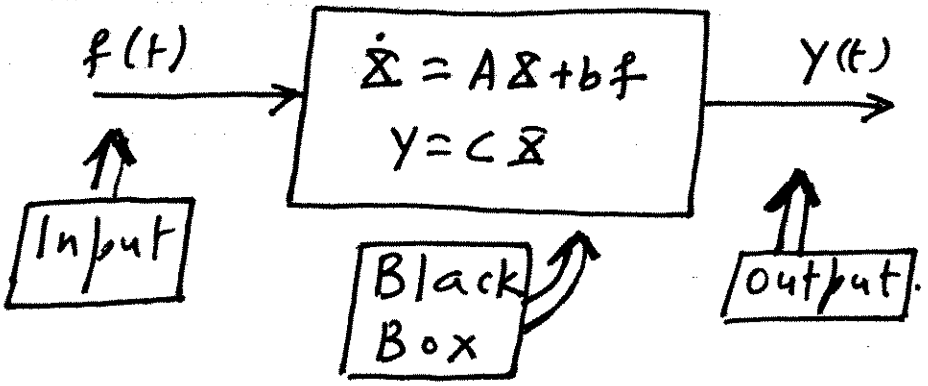
$$=$$

$$Y(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau \,.$$

$\uparrow$

convolution integral, as in page 11·10.

Associated with a differential equation $\dot{\underline{X}} = A\underline{X} + bf$, $y = c\underline{X}$ there is a picture (block diagram)

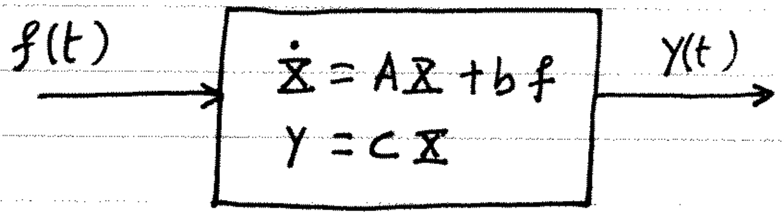


f(t) is the input function. Many often people use u(t) instead.

y(t) is the output function.

$\underline{X}(t)$ is called the state variable.

IV Example:



$h(t) = ce^{At}b$ is given to be

$$e^{-5t} + 3e^{-7t}$$

- Calculate $y(t)$ when $f(t) = \sin 17t$ assume $X(0) = 0$.

Sol$^{\underline{n}}$:

$$y(t) = \int_0^t h(t-\tau)\, f(\tau)\, d\tau$$

$$= \int_0^t \left[e^{-5(t-\tau)} + 3e^{-7(t-\tau)}\right] \sin 17\tau \, d\tau$$

11·16

- Calculate one choice of matrices A, b, c such that

$$h(t) = ce^{At}b = e^{-5t} + 3e^{-7t}$$

Eigenvalues of matrix A are at $-5, -7$.
Char poly of $A = (\lambda+5)(\lambda+7)$

Choose

$$A = \begin{pmatrix} -5 & 0 \\ 0 & -7 \end{pmatrix}, \quad e^{At} = \begin{pmatrix} e^{-5t} & 0 \\ 0 & e^{-7t} \end{pmatrix}$$

Choose

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e^{At}b = \begin{pmatrix} e^{-5t} \\ e^{-7t} \end{pmatrix}$$

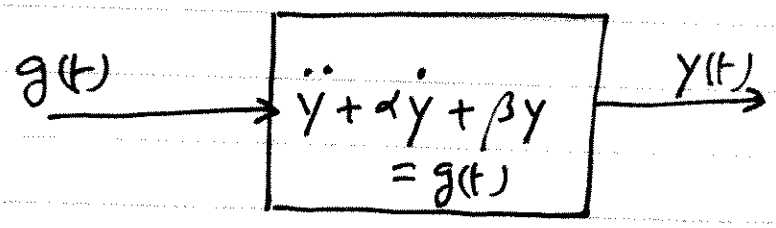
Choose

$$c = \begin{pmatrix} 1 & 3 \end{pmatrix}, \quad ce^{At}b = e^{-5t} + 3e^{-7t}$$

- Write down a differential equation of the form

$$\ddot{y} + \alpha \dot{y} + \beta y = g(t) \quad , \; y(0) = \dot{y}(0) = 0$$

which is equivalent to the black box on page 11·15.



The problem is to calculate α, β and $g(t)$.

sol^{n}

Define $z_1(t) = y(t)$

$z_2(t) = \dot{y}(t)$

It follows that

$$\dot{z}_1 = z_2 , \; \dot{z}_2 = -\alpha z_2 - \beta z_1 + g(t)$$

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix} = \overset{\overset{A_1}{\shortparallel}}{\begin{pmatrix}0 & 1\\ -\beta & -\alpha\end{pmatrix}} \overset{\overset{z}{\shortparallel}}{\begin{pmatrix}z_1\\ z_2\end{pmatrix}} + \overset{\overset{b_1}{\shortparallel}}{\begin{pmatrix}0\\ 1\end{pmatrix}} g(t)$$

$$y(t) = \underset{\underset{c_1}{\shortparallel}}{\begin{pmatrix}1 & 0\end{pmatrix}} \begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

Define $Z(t) = \begin{pmatrix}z_1\\ z_2\end{pmatrix}$, we obtain

$$\left.\begin{aligned}\dot{Z} &= A_1 Z + b_1 g(t)\\ y(t) &= c_1 Z\end{aligned}\right. \qquad \circledast$$

We want to choose A_1 such that $\circledast$ is equivalent to the black box on page 11·15. Also find $g(t)$.

— × —

Eigenvalues of A_1 must be same as the eigenvalues of A which are at $-5, -7$.

characteristic polynomial of A_1 is

$$(\lambda+5)(\lambda+7)$$

$$= \lambda^2 + 12\lambda + 35$$

$$= \lambda^2 + \alpha\lambda + \beta.$$

Hence $\alpha = 12$, $\beta = 35$.

We now need to compute $g(t)$ in terms of $f(t)$.

Note that the diff eqn is

$$\ddot{y} + 12\dot{y} + 35y = g(t). \quad ▨$$

We also have

$$\left.\begin{aligned} \dot{x}_1 &= -5x_1 + f(t) \\ \dot{x}_2 &= -7x_2 + f(t) \\ y(t) &= x_1 + 3x_2 \end{aligned}\right\} \text{follows from page 11·16.} \quad ▨ ▨$$

11.20

We want to write ▨▨ in the form ▨

$$y = x_1 + 3x_2$$

$$\Rightarrow \dot{y} = \dot{x}_1 + 3\dot{x}_2$$

$$= (-5x_1 + f) + 3(-7x_2 + f)$$

$$= -5x_1 - 21x_2 + 4f$$

$$\Rightarrow \ddot{y} = -5(-5x_1 + f) - 21(-7x_2 + f) + 4\dot{f}$$

$$= 25x_1 + 147x_2 + 4\dot{f} - 26f$$

We need to eliminate x_1 & x_2 from here

$$\boxed{\begin{aligned} x_1 + 3x_2 &= y \\ -5x_1 - 21x_2 &= \dot{y} - 4f \end{aligned}}$$

Solving for x_1 & x_2 we get

$$x_1 = \frac{21}{6}y + \frac{1}{2}\dot{y} - 2f; \quad x_2 = -\frac{1}{6}\dot{y} + \frac{2}{3}f - \frac{5}{6}y$$

11·21

We obtain.

$$\ddot{y} = 25\overbrace{\left(\frac{21}{6}y + \frac{1}{2}\dot{y} - 2f\right)}^{x_1}$$

$$+147\underbrace{\left(-\frac{1}{6}\dot{y} + \frac{2}{3}f - \frac{5}{6}y\right)}_{x_2}$$

$$+4\dot{f} - 26f$$

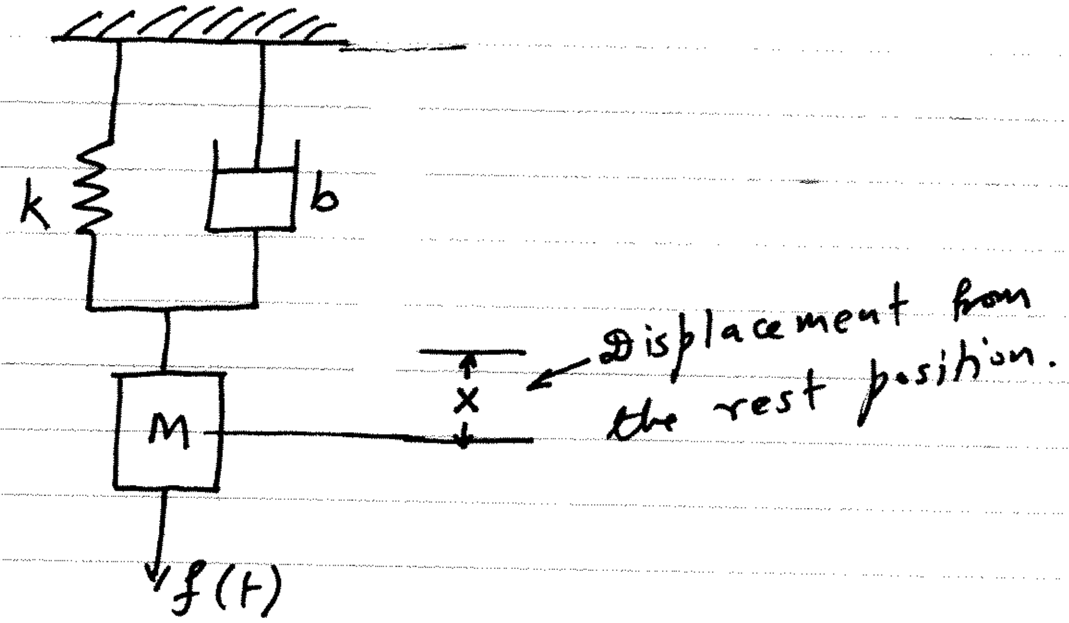
$$\Rightarrow \ddot{y} + 12\dot{y} + 35y = 4\dot{f} - 26f$$

It follows that

$$\boxed{g(t) = 4\frac{df}{dt} - 26f}$$

11·22

V The Mass-spring-Damper story:



$$M\ddot{x} + b\dot{x} + kx = f(t)$$

M: Mass
b: Damping constant.
k: Spring constant

f: Applied force.

Assume $x(0) = 0$, $\dot{x}(0) = 0$, ie the initial position and velocity of the body is zero.

11·23

Define

$$x_1 = x, \quad x_2 = \dot{x}$$

We obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -b/M\,\dot{x} - k/M\,x + 1/M\,f$$

$$= -k/M\,x_1 - b/M\,x_2 + 1/M\,f$$

$$\underline{x} = (x_1 \quad x_2)^T$$

$$\dot{\underline{x}} = A\underline{x} + bf$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1/M \end{pmatrix}$$

Char. poly of A is

$$\lambda^2 + \frac{b}{M}\lambda + \frac{k}{M}$$

Roots are at

$$\frac{-\frac{b}{M} \pm \sqrt{\frac{b^2}{M^2} - \frac{4k}{M}}}{2}$$

(11·24)

$$= -\frac{b}{2M} \pm \sqrt{\left(\frac{b}{2M}\right)^2 - \frac{k}{M}}$$

Case A $b^2 > 4mk$

$$\left(\frac{b}{2M}\right)^2 > \frac{k}{M}$$ Roots λ_1, λ_2 are real and negative

$$\lambda_1 = -\frac{b}{2M} - \sqrt{\left(\frac{b}{2M}\right)^2 - \frac{k}{M}}$$

$$\lambda_2 = -\frac{b}{2M} + \sqrt{\left(\frac{b}{2M}\right)^2 - \frac{k}{M}}$$

The impulse response function $h(t)$ is given by

$$h(t) = \frac{1}{M(\lambda_2 - \lambda_1)}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

$$= \frac{1}{\sqrt{b^2 - 4Mk}}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

11·25

For any applied force $f(t)$, the corresponding $x(t)$ is given by the convolution of $h(t)$ and $f(t)$.

$$x(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau .$$

$$= \frac{1}{\sqrt{b^2-4MK}} \left[\int_0^t e^{-\lambda_2 (t-\tau)} f(\tau)\, d\tau . - \int_0^t e^{-\lambda_1 (t-\tau)} f(\tau)\, d\tau \right]$$

If $f(t) = 1 \quad t > 0$

$\quad\; = 0 \quad t \leq 0$

$$\int_0^t e^{\lambda(t-\tau)} f(\tau) d\tau$$

$$= \frac{1-e^{-\lambda t}}{\lambda}$$

If $f(t) = \sin \omega t \quad t > 0$

$\quad\; = 0 \quad t \leq 0$

$$\int_0^t e^{\lambda(t-\tau)} f(\tau) d\tau$$

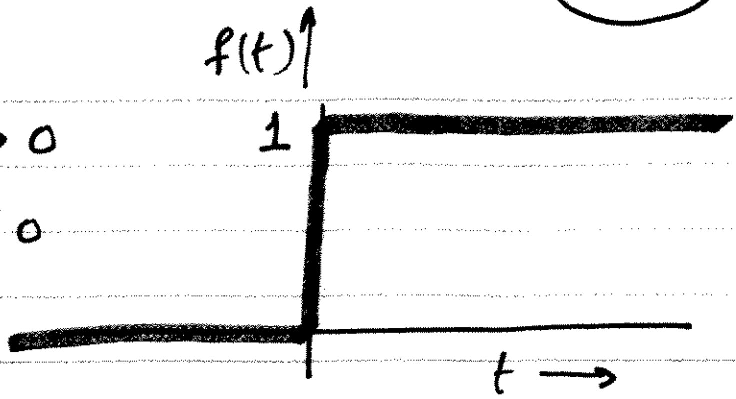
$$= \frac{\lambda \sin \omega t - \omega \cos \omega t + \omega e^{-\lambda t}}{\lambda^2 + \omega^2}$$

Using Matlab

11·26

If $f(t) = 1 \quad t > 0$

$\quad = 0 \quad t \leq 0$



$$x(t) = \frac{1}{\sqrt{b^2 - 4Mk}} \left[\frac{1 - e^{\lambda_1 t}}{\lambda_1} - \frac{1 - e^{\lambda_2 t}}{\lambda_2} \right]$$

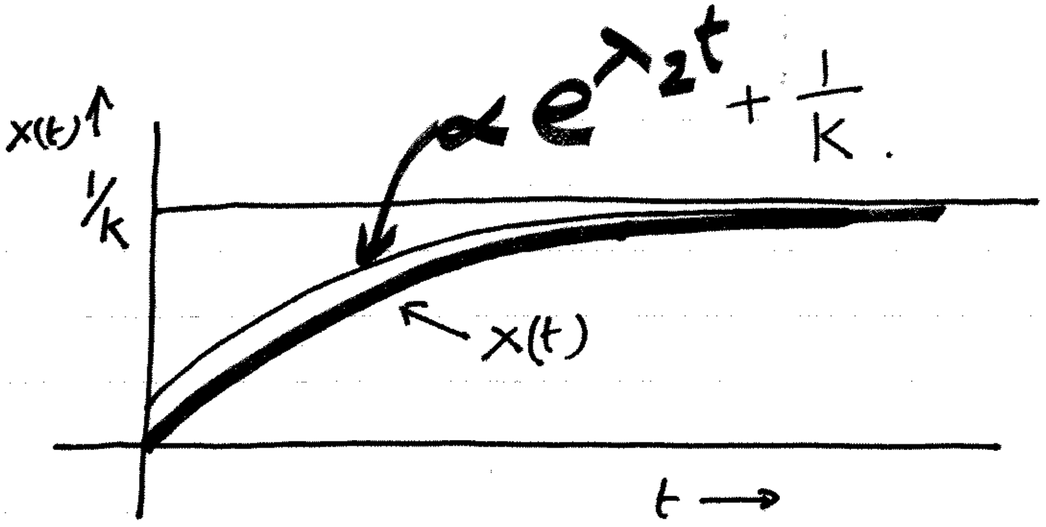
Since λ_1 is more negative than λ_2 $e^{\lambda_1 t}$ goes to zero more rapidly than $e^{\lambda_2 t}$. Thus $x(t)$ approaches the function

$$\frac{1}{\sqrt{b^2 - 4Mk}} \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2} + \frac{e^{\lambda_2 t}}{\lambda_2} \right]$$

$$= \underbrace{\boxed{\frac{1}{M(\lambda_2 - \lambda_1)} \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}}}_{= \frac{1}{k}} + \frac{1}{\sqrt{b^2 - 4Mk}} \frac{e^{\lambda_2 t}}{\lambda_2}$$

(11·27)

$$x(t) \sim \frac{1}{k} + \frac{1}{\sqrt{b^2-4mk}}\,\frac{e^{\lambda_2 t}}{\lambda_2}$$



If $f(t) = \sin\omega t \quad t > 0$

$= 0 \quad t \leq 0$

(11·28)

$$x(t) = \frac{1}{\sqrt{b^2 - 4mk}} \times$$

$$\left[\frac{\omega e^{\lambda_2 t} - \lambda_2 \sin\omega t - \omega\cos\omega t}{\lambda_2^2 + \omega^2} - \frac{\omega e^{\lambda_1 t} - \lambda_1 \sin\omega t - \omega\cos\omega t}{\lambda_1^2 + \omega^2} \right]$$

When t is large b-h $e^{\lambda_1 t}$ & $e^{\lambda_2 t}$ are quite small

$$\lim_{t \to \infty} x(t) =$$

$$\frac{1}{M(\lambda_2 - \lambda_1)} \left[\frac{\lambda_1 \sin\omega t + \omega\cos\omega t}{\lambda_1^2 + \omega^2} - \frac{\lambda_2 \sin\omega t + \omega\cos\omega t}{\lambda_2^2 + \omega^2} \right]$$

$$= \frac{(\lambda_2 + \lambda_1)\omega\cos\omega t + (\lambda_1\lambda_2 + \omega^2)\sin\omega t}{M(\lambda_1^2 + \omega^2)(\lambda_2^2 + \omega^2)}$$

Conclusion:

Under the influence of a periodic forcing function,

$$f(t) = \sin\omega t$$

the displacement $x(t)$ settles down to a periodic function of the form

$$x(t) = A\cos\omega t + B\sin\omega t.$$

where

$$A = \frac{(\lambda_2 + \lambda_1)\,\omega}{M(\lambda_1^2 + \omega^2)(\lambda_2^2 + \omega^2)}$$

$$B = \frac{\lambda_1\lambda_2 + \omega^2}{M(\lambda_1^2 + \omega^2)(\lambda_2^2 + \omega^2)}$$

Case B $b^2 < 4mk$

$$\left(\frac{b}{2M}\right)^2 < \frac{k}{M}$$

Roots λ_1, λ_2 are complex conjugate.

$$\lambda_1 = \overset{\sigma =}{\boxed{-\frac{b}{2M}}} + i\,\boxed{\sqrt{\frac{k}{M} - \left(\frac{b}{2M}\right)^2}} = \omega$$

$$\lambda_2 = -\frac{b}{2M} - i\sqrt{\frac{K}{M} - \left(\frac{b}{2M}\right)^2}$$

The impulse response function $h(t)$ is given by

$$h(t) = \frac{1}{\omega M} e^{\sigma t} \sin \omega t$$

As in page 11·25, for any applied force $f(t)$, the corresponding $x(t)$ is given by the convolution of $h(t)$ and $f(t)$.

$\boxed{11 \cdot 31}$

$$x(t) = \frac{1}{\omega M}\int_0^t e^{\sigma(t-\tau)} \sin[\omega(t-\tau)]\, f(\tau)\, d\tau$$

If $f(t) = 1 \quad t > 0$

$\quad\quad\;\; = 0 \quad t \leq 0$

$$x(t) = \frac{1}{\omega M}\int_0^t e^{\sigma(t-\tau)} \sin[\omega(t-\tau)]\, d\tau$$

$$= \frac{1}{M}\left[\frac{1 + e^{\sigma t}\left[\frac{\sigma}{\omega}\sin\omega t - \cos\omega t\right]}{\sigma^2 + \omega^2}\right]$$

Since $e^{\sigma t} \to 0$ as $t \to \infty$, because $\sigma < 0$

we have

$$\underset{t\to\infty}{\mathrm{Lt}}\; x(t) = \frac{1}{M}\,\frac{1}{\sigma^2+\omega^2} = \frac{1}{k}$$

$$\boxed{\sigma^2 + \omega^2 = \frac{k}{M}}$$

11·32

If $f(t) = \sin \omega_0 t \quad t > 0$
$\quad = 0 \quad t \leqslant 0$

$$x(t) = \frac{1}{\omega M} \int_0^t e^{\sigma(t-\tau)} \sin \omega(t-\tau) \sin \omega_0 \tau \, d\tau$$

$$= \frac{1}{\omega M} \times$$

$$\frac{[(2\sigma\omega^3\omega_0)\ cos\omega_0 t + (\omega^2 + \omega^3\sigma^2 - \omega_0^2)\ sin\omega_0 t]\ \omega - (\omega_0 e^{\sigma t})[2\omega\sigma cos\omega t + (\sigma^2 + \omega_0^2 - \omega^2) sin\omega t]}{[\sigma^2 + (\omega + \omega_0)^2][\sigma^2 + (\omega - \omega_0)^2]}$$

$$\underset{t \to \infty}{Lt}\ x(t) = \frac{2\sigma\omega^3\omega_0 \cos\omega_0 t + (\omega^2 + \omega^3\sigma^2 - \omega_0^2) \sin\omega_0 t}{M[\sigma^2 + (\omega+\omega_0)^2][\sigma^2 + (\omega - \omega_0)^2]}$$

once again because $\sigma < 0$ and $e^{\sigma t} \to 0$ as $t \to \infty$.

Conclusion:

Under the influence of a periodic forcing function,

$$f(t) = \sin \omega_0 t$$

the displacement $x(t)$ settles down to a periodic function of the form

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t.$$

11·34

Forced oscillations & Resonance

Assume that the frequency ω_0 of the periodic forcing function is equal to ω ie

$$f(t) = \left.\begin{matrix} \sin\omega t & t > 0 \\ 0 & t \leq 0 \end{matrix}\right\} \text{ where}$$

$$\omega = \sqrt{\frac{k}{M} - \left(\frac{b}{2M}\right)^2}$$

The displacement $x(t)$ is given by

$$x(t) = \frac{1}{\omega M}\int_0^t e^{\sigma(t-\tau)} \sin\omega(t-\tau) \sin\omega\tau \, d\tau.$$

$$= \frac{\sigma \sin\omega t\left(1+e^{\sigma t}\right) + 2\omega \cos\omega t\left(1-e^{\sigma t}\right)}{M\sigma\left(\sigma^2 + 4\omega^2\right)}$$

For small values of σ, which would be the case if the damping constant b is small, we have

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$$x(t) = \frac{\cancel{2}\cancel{\omega} \cos \omega t}{M\sigma \underset{2}{\cancel{4}} \omega^{\cancel{2}}}$$

$$= \frac{1}{2M\omega} \frac{\cos \omega t}{\sigma}$$

Note that the amplitude of the function $x(t)$ increases with smaller σ.

<u>Resonance:</u>

Resonance occurs when $\sigma = 0$, $\omega = \omega_0$. In this case

$$x(t) = \frac{1}{\omega M} \int_0^t \sin \omega (t - \tau) \sin \omega \tau \, d\tau .$$

$$= \frac{1}{2M\omega} \left[\frac{\sin \omega t}{\omega} - t \cos \omega t \right]$$

which is unbounded as a result of 't' in the amplitude of $\cos \omega t$.